Math 210A Lecture 3 Notes

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1 The Yoneda Lemma

1.1 Two versions of the Yoneda lemma

Lemma 1.1 (Yoneda)**.** *Let $\mathcal{C}$ be a small category, and let $h^{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ be $h^{\mathcal{C}}(A) = h^A = \mathrm{Hom}_{\mathcal{C}}(\cdot, A)$ and if $f : A \to B$, then $h^{\mathcal{C}}(f)_X : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ sends $(g : X \to A) \mapsto (f \circ g : X \to B)$. Then $h^{\mathcal{C}}$ is fully faithful.*

Proof. To show that $h^{\mathcal{C}}$ is faithful, let $f, g : A \to B$, and suppose that $h^{\mathcal{C}}(f) = h^{\mathcal{C}}(g)$. THen $h^{\mathcal{C}}(f)A, h^{\mathcal{C}}(g)_A : \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, B)$ maps $1_A \mapsto f \circ 1_A = f$ and $1_A \mapsto g \circ 1_A = g$. So $f = g$.

To show that $h^{\mathcal{C}}$ is full, let $\{\eta_X\} : h^A \to h^B$. We claim that $h^{\mathcal{C}}(\eta_A(1_A)) = \eta$.

$$\begin{tikzcd} h^A(A) \arrow[r, "\eta_A"] \arrow[d, "h^A(f)"] & h^B(A) \arrow[d, "h^B(f)"] \\ h^A(C) \arrow[r, "\eta_C"] & h^B(C) \end{tikzcd}$$

This is

$$\begin{tikzcd} \mathrm{Hom}(A, A) \arrow[r, "\eta_A"] \arrow[d, "h^A(f)"] & \mathrm{Hom}(B, B) \arrow[d, "h^B(f)"] \\ \mathrm{Hom}(C, A) \arrow[r, "\eta_C"] & \mathrm{Hom}(C, B). \end{tikzcd}$$

Since this diagram commutes, $\eta_C \circ h^A(f) = h^B(f) \circ \eta_A$. So they are equal on evaution on an element. Then $\eta_C \circ h^A(f)[1_A] = h^B(f) \circ \eta_A[1_A]$, so $\eta_C[f] = \eta_A[1_A] \circ f$. In particular, $\eta = h^{\mathcal{C}}(\eta_A[1_A])$. $\square$

Lemma 1.2 (Yoneda, strengthened)**.** *Let $\mathcal{C}$ be a small category, let $h^{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ be the Yoneda embedding, and let $F : \mathcal{C}^{op} \to \mathrm{Set}$. Then $\mathrm{Nat}(h^A, F)$ is in bijection with $F(A)$.*

Proof. Define $\Phi : \mathrm{Nat}(h^A, F) \to F(A)$ given by $\eta_A : h^A(A) \to F(A)$, which sends $1_A \mapsto \eta_A(1_A)$. Define $\Psi : F(A) \to \mathrm{Nat}(h^A, F)$. Then, for $x \in F(A)$, $\Psi(x)_B : h^A(B) = \mathrm{Hom}(B, A) \to F(B)$ is $\Psi(x) = \mathrm{ev}_x \circ F$.

We claim that $\Phi \circ \Psi$ is the identity on $F(A)$. Let $x \in F(A)$. Then $\Phi(\Psi(x)) = \Phi(\mathrm{ev}_x \circ F) = \mathrm{ev}_x \circ 1_{F(A)} = x$. $(\Psi \circ \Phi)(\eta) = \Psi(\eta_A(1_A)) = \mathrm{ev}_{\eta_A(1_A)} \circ F$. Let $f : B \to A$. Then

$$\begin{array}{ccc} \mathrm{Hom}(A, A) & \xrightarrow{\eta_A} & F(A) \\ \Big\downarrow{\scriptstyle h^A(f)} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(B, A) & \xrightarrow{\eta_B} & F(B). \end{array}$$

So $F(f) \circ \eta_A = \eta_B \circ h^A(f)$, which means $F(f) \circ \eta_A(1_A) = \eta_B \circ h^A(f)(1_A)$. The left hand side is $\Phi \circ \Phi(\eta)_B[f]$, and the right hand side is $\eta_B(f)$. Therefore, $\Psi \circ \Phi(\eta) = \eta$. $\square$

This form of the Yoneda lemma implies the previous version.

Corollary 1.1 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a small category, and let $h^{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$. Then $h^{\mathcal{C}}$ is fully faithful.*

Proof. Let $B \in \mathrm{Obj}(\mathcal{C})$. Consider $F = h^B - \mathrm{Hom}_{\mathcal{C}}(\cdot, B)$. Then $\mathrm{Nat}(h^A, h^B)$ is in bijection (via F) with $h^B(A) = \mathrm{Hom}_{\mathcal{C}}(A, B)$. $\square$

1.2 Partially ordered sets

Definition 1.1. A **partially ordered set (poset)** is a set S with a relation $\leq$ on S such that

1. $x \leq x$ for all $x \in S$,
2. if $x \leq y$ and $y \leq x$, then $x = y$,
3. if $x \leq y$ and $y \leq z$, then $x \leq z$.

We can turn a poset into a category. Let $\mathrm{Obj}(\mathcal{C}_S) = S$ and

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \begin{cases} \{\text{unique morphism}\} & x \leq y \\ \varnothing & \text{otherwise.} \end{cases}$$